CONVERGENCE RATES IN THE LAW OF LARGE NUMBERS FOR $B$-VALUED RANDOM ELEMENTS

Liang Hanying (梁汉营)  Wang Li (王丽)
Department of Applied Mathematics, Tongji University, Shanghai 200092, China

Abstract The author discusses necessary and sufficient conditions of the complete convergence for sums of $B$-valued independent but not necessarily identically distributed r.v.'s in Banach space of type $p$, and obtain characterization of Banach space of type $p$ in terms of the complete convergence. A series of classical results on iid real valued r.v.'s are extended. As application author gives the analogous results for randomly indexed sums.

Key words Convergence rate, random element, Banach space of type $p$, slowly varying function

1991 MR Subject Classification  60B12

1 Introduction and Main Results

Let $\{X_n, n \geq 1\}$ be a sequence of random variables in the same probability space and put $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. In real-valued case, the rates of convergence to zero of quantities $P(|S_n|/n^{1/t} > \epsilon)$ ($0 < t < 2$) are described by the well-known theorem of Baum and Katz (1965):

Theorem BK Let $0 < t < 2, r \geq 1$, and let $\{X_n, n \geq 1\}$ be real valued iid r.v.'s, when $1 \leq t < 2$, $EX_1 = 0$. Then $\sum_{n=1}^{\infty} n^{r-2}P(|S_n| \geq \epsilon \cdot n^{1/t}) < +\infty \quad \forall \epsilon > 0 \iff E|X_1|^{rt} < +\infty$.

In 1985, Bai and Su generalized Theorem BK and obtained the corresponding results of randomly indexed sums. The aim of this paper is to extend the results of Bai and Su (1985) to not necessarily identically distributed r.v.'s in Banach space; to find its necessary and sufficient conditions and give characterization of geometric properties, type $p$, of Banach spaces. In the sequel, let $B$ be a real separable Banach space with norm $\| \cdot \|$.

Write $\{X_n\} \ll X$ if there exists a constant $C > 0$ such that, for sufficiently large $x, n$,

$$\sum_{i=1}^{n} P(\|X_i\| > x) \leq CNP(|X| > x), \text{ where } X \text{ is a real random variable.}$$

Write $\{X_n\} \gg X$ if there exists a constant $C > 0$ such that, for sufficiently large $x, n$,

$$\sum_{i=1}^{n} P(\|X_i\| > x) \geq CNP(|X| > x), \text{ where } X \text{ is a real random variable.}$$

1Received July 13,1999. Supported by the Science Fund of Tongji University
A Banach space $B$ is called type $p$ ($1 \leq p \leq 2$) if for any zero mean $B$-valued independent random element sequence $\{X_n, n \geq 1\}$, there exists $C = C_p > 0$ such that

$$E\|\sum_{i=1}^{n} X_i\|^p \leq C \sum_{i=1}^{n} E\|X_i\|^p, n \geq 1.$$ 

Our main results are as follows:

**Theorem 1** Let $1 \leq t < 2$. Suppose that $l(x) > 0$ be a slowly varying function as $x \to +\infty$. The following statements are equivalent:

(a) $B$ is of type $p$ for some $t < p \leq 2$;

(b) Suppose $r > 1$. For each sequence $\{X_n\}$ of zero mean $B$-valued independent random elements, if $\{X_n\} \ll X, E[|X|^t l(|X|^t)] < +\infty,$ (1.1) then $\forall \epsilon > 0$ we have

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_n\| \geq \epsilon \cdot n^{1/t}) < +\infty,$$ (1.2)

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon \cdot n^{1/t}) < +\infty,$$ (1.3)

$$\sum_{n=1}^{\infty} n^{r-2} l(n) P(\sup_{k \geq n} \|S_k/k^{1/t}\| \geq \epsilon) < +\infty.$$ (1.4)

(c) Suppose that $l(x)$ is non-decreasing. For each sequence $\{X_n\}$ of zero mean $B$-valued independent random elements, if $\{X_n\} \ll X, E[|X|^t l(|X|^t)] < +\infty,$ (1.5) then $\forall \epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\|S_n\| \geq \epsilon \cdot n^{1/t}) < +\infty,$$ (1.6)

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon \cdot n^{1/t}) < +\infty.$$ (1.7)

(d) Suppose that $l(x)$ satisfies, for every positive integer $k$

$$\sum_{i=1}^{k} l(2^i) \leq C \cdot k \cdot l(2^k).$$ (1.8)

For each sequence $\{X_n\}$ of zero mean $B$-valued independent random elements, if $\{X_n\} \ll X, E[|X|^t l(|X|^t) \log^+ |X|] < +\infty,$ (1.9) then $\forall \epsilon > 0$ we have

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\sup_{k \geq n} \|S_k/k^{1/t}\| \geq \epsilon) < +\infty.$$ (1.10)

**Theorem 2** Let $0 < t < 2$, $l(x) > 0$ be a slowly varying function as $x \to +\infty$, and let $\{X_n\}$ be a sequence of zero mean $B$-valued independent random elements.
(a) Suppose \( r > 1 \) and \( \{X_n\} \gg X \). If one of the sums holds in (1.2), (1.3) and (1.4), then \( E[|X|^r l(|X|^r)] < +\infty \).

(b) Suppose \( \{X_n\} \gg X \) and \( l(x) \) is a non-decreasing. If one of the sums holds in (1.6) and (1.7), then \( E[|X|^r l(|X|^r)] < +\infty \).

(c) Suppose that \( \{X_n\} \) is iid r.v.’s and that \( l(x) \) satisfies \( \sum_{i=1}^{k} l(2^i) \geq C \cdot k \cdot l(2^k) \) for every positive integer \( k \) and \( l(n) \log n \geq \delta > 0, n \geq n_0 \). Then (1.10) implies

\[
E[||X||^r l(||X||^r) \log^+ ||X||] < +\infty.
\]

**Theorem 3** Let \( 0 < t < 1, l(x) > 0 \) be a slowly varying function as \( x \to +\infty \), and let \( \{X_n\} \) be any sequence of B-valued random elements.

(a) Suppose \( r > 1 \). If \( \{X_n\} \) is independent and (1.1) is satisfied, then (1.2), (1.3) and (1.4) are equivalent and hold.

(b) Suppose \( l(x) \) is non-decreasing. If (1.5) is satisfied, then (1.6) and (1.7) are equivalent and hold.

(c) Suppose that \( l(x) \) satisfies (1.8). Then (1.9) implies (1.10).

**Corollary 1** (Hu, M´oricz and Taylor (1989), Theorem 2) Let \( \{X_{nk}\} \) be an array of rowwise independent real-valued r.v.’s with \( EX_{nk} = 0 \) and \( P(|X_{nk}| > x) \leq P(|X| > x) \) for all \( n, k \) and \( x > 0 \). If \( E|X|^2 < \infty \) for some \( 1 \leq t < 2 \), then

\[
\frac{1}{n^{2/t}} \sum_{k=1}^{n} X_{nk} \to 0 \text{ completely, i.e. } \sum_{n=1}^{\infty} P(\|\sum_{k=1}^{n} X_{nk}\| \geq \epsilon n^{1/t}) < \infty, \forall \epsilon > 0.
\]

In this section, let \( \{\tau_n, n \geq 1\} \) be a sequence of non-negative, integer valued random variables and \( \tau \) a positive random variable. All random variables are defined on the same probability space.

**Theorem 4** Let \( 1 \leq t < 2 \). Suppose that \( l(x) > 0 \) is a slowly varying function as \( x \to +\infty \). The following statements are equivalent:

(a) \( B \) is of type \( p \) for some \( t < p \leq 2 \);

(b) Under the assumptions of Theorem 1(b) and (d), respectively, if there exists \( \gamma > 0 \) such that \( \sum_{n=1}^{\infty} n^{r-2} l(n) P(\frac{\tau_n}{n} < \gamma) < +\infty \). Then \( \forall \epsilon > 0, \)

\[
\sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_{\tau_n}\| \geq \epsilon \cdot \tau_n^{1/t}) < +\infty.
\]

(c) Under the assumptions of Theorem 1(b)-(d), respectively, if there exists \( \epsilon_0 > 0 \) such that \( \sum_{n=1}^{\infty} n^{r-2} l(n) P(\|\frac{\tau_n}{n} - \tau\| > \epsilon_0) < +\infty \), with \( P(\tau \leq B) = 1 \) for some \( B > 0 \), then \( \forall \epsilon > 0, \)

\[
\sum_{n=1}^{\infty} n^{r-2} l(n) P(\|S_{\tau_n}\| \geq \epsilon \cdot n^{1/t}) < +\infty.
\]

**Remark** Let \( r > 1 \), and let \( \{X_n\} \) be a sequence of iid real valued r.v.’s. Assume that \( E|X|^r < +\infty \), and for any given \( \epsilon > 0, \)

\[
\sum_{n=1}^{\infty} n^{r-2} P(\|\frac{\tau_n}{n} - \tau\| > \epsilon) < +\infty, \text{ where } P(A \leq \tau \leq
\]
Thus, to prove $B = 1$ for some real numbers $A, B$, such that $0 < A < B < +\infty$, Lin (1986) proved that

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\sum_{k=1}^{\tau_n} (X_k - EX_k) \right) \geq \epsilon \cdot \tau_n < +\infty.$$ 

It is easy to show that if $P(A \leq \tau \leq B) = 1$ for some real number $A, B$, where $0 < A < B < +\infty$ or $P(A \leq \tau) = 1$ for some $A > 0$, then for any given $\epsilon > 0(\epsilon < A)$, $P\left(\frac{\tau_n}{n} < A - \epsilon \right) \leq P\left(\frac{\tau_n}{n} - \tau \right) > \epsilon$. So, the above result of Lin is a special case of our results in real space.

In the sequel, $C$ denotes a positive constant whose value may change at different place.

### 2 Proofs of Main Results

It is well-known that if $l(x) > 0$ is a slowly varying function as $x \to +\infty$, then

1. $\lim_{x \to +\infty} \frac{l(tx)}{l(x)} = 1$, $\forall t > 0$;
2. $\lim_{k \to +\infty} \sup_{2^k \leq x \leq 2^{k+1}} \frac{l(x)}{l(2^k)} = 1.$
3. $\lim_{x \to +\infty} x^\delta l(x) = +\infty$, $\lim_{x \to +\infty} x^{-\delta} l(x) = 0$, $\forall \delta > 0$.

**Lemma 1** (Shao (1988), Corollary 2.4) Let $\{X_n\}$ be a sequence of zero mean independent random elements in a type $p$ space $B$. Then for $q \geq p$, there exists a positive constant $C = C_p$ such that $E \max_{1 \leq k \leq n} |S_k|^q \leq (96q^q \{C \sum_{i=1}^n E\|X_i\|^p\}^{q/p} + E \max_{1 \leq k \leq n} \|X_k\|^q\}.$

**Proof of Theorem 1** (a)$\Rightarrow$(b) Clearly, we need only to prove (1.3) and (1.4).

Firstly, we prove (1.3). Let $Y_{ij} = X_j I(||X_j|| \leq 2^j/t)$, $S_{ik} = \sum_{j=1}^k Y_{ij}$, $T_{ik} = S_k - S_{ik}$, then by the properties of $l(x)$ we have

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon \cdot n^{1/t}\right) \leq C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) P\left(\max_{1 \leq k \leq 2^{i+1}} \|T_{ik}\| \geq \epsilon \cdot 2^{i/t}\right)$$

$$+C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) P\left(\max_{1 \leq k \leq 2^{i+1}} \|S_{ik}\| \geq \epsilon \cdot 2^{i/t}\right) + C =: I_1 + I_2 + C.$$

Note that, from Lemma 1 of Bai and Su (1985) we have $\sum_{i=0}^{\infty} 2^i l(2^i) P(\|X\| > 2^{i/t}) < \inftyock E[|X|^t l(|X|^t)] < +\infty$, which follows $I_1 \leq C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \sum_{j=1}^{2^{i+1}} P(\|X_j\| > 2^j/t) < +\infty$. For $t < s < rt$, by the property 3 of $l(x)$, the condition (1.1) implies $E[|X|^s] < +\infty$, hence, from $EX_n = 0$ we have

$$\max_{1 \leq k \leq 2^{i+1}} \|ES_{ik}\|/2^{i/t} \leq C 2^{(1-s/t)i} E|X|^s [1 + \int_{1}^{+\infty} y^{s-1} dy] \to 0. \quad (2.1)$$

Thus, to prove $I_2 < \infty$, it is sufficient to show that

$$I_2 = \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) P(\max_{1 \leq k \leq 2^{i+1}} \|S_{ik} - ES_{ik}\| \geq \epsilon \cdot 2^{i/t}) < +\infty, \quad \forall \epsilon > 0.$$

In fact, choosing $q > \max\{p, \frac{pt(r-1)}{p-1}, rt\}$, and using Lemma 1 we have

$$I_2 \leq C \sum_{i=0}^{\infty} 2^{i(r-1-\eta/t)} l(2^i) \left\{\sum_{j=1}^{2^{i+1}} E\|Y_{ij}\|^p 2^{i+1} + \sum_{j=1}^{2^{i+1}} E\|Y_{ij}\|^q\right\} =: I_3 + I_4.$$
If \( rt \leq p \),

\[
I_3 \leq C \sum_{i=0}^{\infty} 2^{i(r-1-q/t)} l(2^i) \int_0^{2^{i+1}} x^{p/t-1} P(\|X_j\|^t > x)dx |^{q/p} \ \\
\leq C \sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) + C \sum_{i=0}^{\infty} \sum_{k=k_0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) \int_{2^{i-1}}^{2^i} x^{p/t-1} P(\|X_t\|^t > x)dx |^{q/p} \ \\
\leq C + C \sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) \sum_{k=k_0}^{\infty} 2^{kr} l(2^k) P(\|X_t\|^t > 2^{k-1}) l(2^i) (1 + \delta)^{\frac{q}{p}} (l(2^i))^{-\frac{q}{p}} \ \\
\leq C + C \sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) 1^{1-q/p} (1 + \delta)^{q/p} < +\infty,
\]

where \( \delta > 0 \) is small enough. Similarly, if \( rt > p \),

\[
I_3 \leq C \sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) \ \\
+ C \sum_{i=0}^{\infty} 2^{i(r-1-q/t+q/p)} l(2^i) 1^{1-q/p} (1 + \delta)^{q/p} \{E[|X_t|^{t}(|X_t|^t)]\}^{q/p} < +\infty.
\]

As in the above proof we get \( I_4 \leq C \sum_{i=0}^{\infty} 2^{i(r-q/t)} l(2^i) + C \sum_{k=1}^{\infty} 2^{kr} l(2^k) P(\|X_t\|^t > 2^{k-1}) < +\infty \).

Next, we prove (1.4). By the property of \( l(x) \) and Lemma 1 of Bai and Su (1985) we have

\[
\sum_{n=1}^{\infty} n^{r-2} l(n) P(\sup_{k \geq n} \|S_k/k^{1/t}\| \geq \epsilon) \leq C \sum_{i=0}^{\infty} 2^i \int_{2^i}^{2^{i+1}} P(\sup_{k \geq 2^i} \|S_k/k^{1/t}\| \geq \epsilon) + C \ \\
\leq C \sum_{m=0}^{\infty} 2^m (2^i P(\max_{2^m \leq k < 2^{m+1}} \|S_k\| \geq \epsilon \cdot 2^m) + C.
\]

Similarly to the proof of (1.3), (1.4) is proved.

(a)\( \Rightarrow \) (c) Clearly, we only need to prove (1.7). Let \( Y_{ij}, S_{ik}, T_{ik} \) be as in (a)\( \Rightarrow \) (b). By the properties of \( l(x) \) we have

\[
\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon \cdot n^{1/t}) \leq C \sum_{i=0}^{\infty} l(2^i) P(\max_{1 \leq k < 2^{i+1}} \|T_{ik}\| \geq \epsilon \cdot 2^i) \ \\
+ C \sum_{i=0}^{\infty} l(2^i) P(\max_{1 \leq k < 2^{i+1}} \|S_k\| \geq \epsilon \cdot 2^i) + C =: I_5 + I_6 + C.
\]

From \( E[|X_t|^{t} l(|X_t|^t)] < +\infty \), it is easy to prove \( I_5 < +\infty \).

Since \( l(x) \) is a positive non-decreasing function, the condition (1.5) implies \( E|X|^t < +\infty \).

By \( EX_n = 0 \) we get

\[
\max_{1 \leq k < 2^{i+1}} \|ES_{ik}\|^t / 2^{i/t} \leq C 2^i P(\|X_t\|^t > 2^i) + C \sum_{k=i}^{\infty} 2^k P(\|X_t\|^t > 2^k) \to 0.
\]

Thus, to prove \( I_6 < \infty \), it suffices to show that

\[
I_6 =: \sum_{i=0}^{\infty} l(2^i) P(\max_{1 \leq k < 2^{i+1}} \|S_{ik} - ES_{ik}\| \geq \epsilon \cdot 2^i) < +\infty, \quad \forall \epsilon > 0.
\]
In fact, from the definition of type $p$ and similarly to the proof of $I_4$ we get

$$I_6^* \leq \sum_{i=0}^{\infty} l(2^i)2^{n(1-p/t)} + CE[X^t l(|X|^t)] < +\infty.$$  

(a)⇒(d) By the properties of $l(x)$ and (1.8), similarly to the proof of (a)⇒(c) we get

$$\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\sup_{k \geq n} \|S_k/k^{1/t}\| \geq \epsilon) \leq C \sum_{m=1}^{\infty} ml(2^m)P(\max_{2^m \leq k < 2^{m+1}} \|S_k\| \geq \epsilon \cdot 2^{m/t}) + C < \infty.$$  

(b)⇒(a), (c)⇒(a) and (d)⇒(a) We only prove (b)⇒(a), the proof of (c)⇒(a) and (d)⇒(a) is analogous. Suppose that (b) is satisfied. In (b), take $l(x) = 1$, and further let $\{X_n\}$ be independent and symmetric. Note that (1.3) and (1.4) imply (1.2). Thus, if $E|X|^t < +\infty$, then we have

$$\sum_{n=1}^{\infty} n^{r-2}P(\|S_n\| \geq \epsilon \cdot n^{1/t}) < +\infty, \quad \forall \epsilon > 0,$$

which implies from $r > 1$ that

$$S_n/n^{1/t} \rightarrow 0 \quad a.s.$$  

(2.2)

By (2.2), similarly to the proof of Theorem 1 in Liang and Niu (1998), we can verify that $B$ is of type $p$ for $t < p \leq 2$.

Proof of Theorem 2 (a) It suffices to show that (1.2)⇒$E|X|^t l(|X|^t)] < +\infty$. Define $X_n^* = X_n - X_0^*$, where $X_0^*$ denotes independent copy of $X_n$ and set $S_n^* = \sum_{i=1}^{n} X_n^i$. Then $\{X_n^*\}$ is a sequence of independent symmetric $B$-valued r.v.’s, and

$$\sum_{n=1}^{\infty} n^{r-2}l(n)P(\|S_n^*\| \geq \epsilon \cdot n^{1/t}) < +\infty, \quad \forall \epsilon > 0,$$

which implies from $r > 1$ that

$$S_n^*/n^{1/t} \xrightarrow{P} 0.$$  

(2.3)

By using (2.3) and (2.4) we can get

$$\sum_{k=1}^{n} P(\|X_k^1\| \geq 2n^{1/t}) \leq 4P(\|S_n^*\| \geq n^{1/t}), \quad n \geq n_1$$  

(2.5)

and choose $n \geq n_2$ such that $P(\|S_n^*\| \geq n^{1/t}) < \frac{1}{8^t}$. Thus, by Proposition 6.8 of Ledoux and Talagrand (1991) we get $E\|S_n^*\|^2/n^{1/t} < \infty$. While using $EX_n = 0$ we get $E\|S_n\|^2/n^{1/t} \leq E\|S_n^*\|^2/n^{1/t}$. Therefore, we have

$$S_n/n^{1/t} \xrightarrow{P} 0.$$  

(2.6)

By (2.6), and using the Ottaviani inequality (cf: Wu and Wang (1990), P.15), we get $P(\|X_0\| \geq n^{1/t}) \leq P(\max_{1 \leq k \leq n} \|S_k\| \geq \frac{1}{2}n^{1/t}) \leq CP(\|S_n\| \geq \frac{1}{4}n^{1/t}) \leq \frac{1}{2}$ for $n$ sufficiently large, and hence by symmetrization inequality (cf: Wu and Wang (1990), P.114) and (2.5)

$$nP(|X| \geq n^{1/t}) \leq C \sum_{k=1}^{n} P(\|X_k\| \geq n^{1/t}) \leq C \sum_{k=1}^{n} P(\|X_k^1\| \geq \frac{1}{2}n^{1/t}) \leq CP(\|S_n^*\| \geq \frac{1}{4}n^{1/t})$$

which follows from (2.3) that $\sum_{n=1}^{\infty} n^{r-2}l(n)P(|X| \geq n^{1/t}) < +\infty$, hence, by Lemma 1 of Bai and Su (1985), we get $E[|X|^t l(|X|^t)] < +\infty$. 


We can similarly prove (b). Next, we prove (c). Note that for every \( j \geq 1 \),
\[
\bigcup_{i=j}^{\infty} \bigcup_{k=2^i}^{2^{i+1}-1} \{\|S_k - S_{2^i}\| \geq 2 \epsilon \cdot 2^{i/t}\} \subset \bigcup_{i=j-1}^{\infty} \bigcup_{k=2^i}^{2^{i+1}-1} \{\|S_k\| \geq \epsilon \cdot 2^{i/t}\},
\]
and hence \( P(\sup_{i \geq j} \max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon) \leq P(\sup_{i \geq j-1} \max_{2^i \leq k < 2^{i+1}} \|S_k\| / 2^{i/t} \geq \epsilon) \). Suppose that
\[
\sum_{i=1}^{\infty} P(\max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon) = \infty,
\]
then from Borel-Cantelli lemma and noticing that \( \{\max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t}, i \geq 1\} \) is independent, we have \( P(\max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon, \ i.o.) = 1 \) and
\[
P(\sup_{i \geq j} \max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon) = 1, \ \forall j \geq 1.
\]

While, from (2.8)
\[
\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\sup_{k \geq n} \|S_k / k^{1/t}\| \geq \epsilon \cdot 2^{-1/t}) \geq C \sum_{j=1}^{\infty} l(2^j) P(\sup_{i \geq j} \max_{2^i \leq k < 2^{i+1}} \|S_k\| / 2^{i/t} \geq 2 \epsilon)
\]
\[
= C \sum_{j=1}^{\infty} l(2^j) = \infty,
\]
which is in contradiction with the assumption.

So, \( \sum_{i=1}^{\infty} P(\max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon) < \infty \), which implies
\[
P(\sup_{i \geq j} \max_{2^i \leq k < 2^{i+1}} \|S_k - S_{2^i}\| / 2^{i/t} \geq 2 \epsilon) = 2 \sum_{i=j}^{\infty} P(\max_{1 \leq k < 2^i} \|S_k\| \geq 2 \epsilon \cdot 2^{i/t}).
\]

Therefore
\[
\sum_{n=1}^{\infty} \frac{l(n)}{n} P(\sup_{k \geq n} \|S_k / k^{1/t}\| \geq \epsilon \cdot 2^{-1/t}) \geq C \sum_{n=1}^{\infty} \frac{l(n) \log n}{n} P(\max_{1 \leq k \leq n} \|S_k\| \geq C \epsilon \cdot n^{1/t}).
\]

The proof of the rest is similar to that in (b).

**Proof of Theorem 3**  
(a) By the property 3 of \( l(x) \), we know that the condition \( E[|X|^t l(|X|^t)] < +\infty \) implies \( E|X|^t < +\infty \), it is easy to verify
\[
S_n / n^{1/t} \xrightarrow{P} 0.
\]

By (2.9) and using the Ottaviani inequality we have
\[
P(\max_{1 \leq k \leq n} \|S_k\| \geq \epsilon \cdot n^{1/t}) \leq CP(\|S_n\| \geq \frac{\epsilon}{2} \cdot n^{1/t})
\]
for \( n \) sufficiently large. \( \forall \epsilon > 0 \), when \( 2^m < n \leq 2^{m+1} \) for \( m \) sufficiently large, using the Ottaviani inequality we can obtain that
\[
P(\|S_{2^m}\| \geq \epsilon \cdot 2^{m/t}) \leq CP(\|S_n\| \geq C \epsilon \cdot n^{1/t}).
\]
By the property of \( l(x) \), and using Lemma 1 of Bai and Su (1985) and (2.11) we have
\[
\sum_{n=1}^{\infty} n^{-2} l(n) P(\sup_{k \geq n} \| S_k / k^{1/t} \| \geq \epsilon) \leq C \sum_{n=1}^{\infty} n^{-2} l(n) P(\| S_n \| \geq C \epsilon \cdot n^{1/t}) + C. \tag{2.12}
\]
Hence, by (2.10) and (2.12) we obtain that (1.2), (1.3) and (1.4) are equivalent. Thus, we only need to prove that (1.2) holds. By symmetrization inequality, we may suppose that \( \{ X_n \} \) is symmetric. Let \( Y_{ni} = X_i I(\| X_i \| \leq n^{1/t}) \), \( S'_n = \sum_{i=1}^{n} Y_{ni} \), \( S''_n = S_n - S'_n \), obviously
\[
\sum_{n=1}^{\infty} n^{-2} l(n) P(\| S_n \| \geq \epsilon \cdot n^{1/t}) \leq C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i \leq n < 2^{i+1}} P(\| S'_n \| \geq \frac{\epsilon}{2} \cdot n^{1/t}) + C =: J_1 + J_2 + C.
\]
Similarly to the proof of \( I_1 \) we can get \( J_1 < \infty \) and \( S''_n / n^{1/t} \overset{P}{\to} 0 \), further we obtain \( S''_n / n^{1/t} \overset{P}{\to} 0 \) from (2.9), and using Lemma 3.1 of De Acosta (1981) we have \( E \| S'_n / n^{1/t} \| \to 0 \), as \( n \to +\infty \). Thus, to prove \( J_2 < \infty \), it suffices to show that
\[
J_2 = \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i \leq n < 2^{i+1}} P(\| S'_n \| - E \| S'_n \| \geq \epsilon \cdot n^{1/t}) < +\infty, \quad \forall \epsilon > 0.
\]
In fact, if \( q > \max \{ 2, \frac{2t(r-1)}{2-rt} \} \), using Theorem 2.1 of De Acosta (1981) and as in the proof of \( I_2 \) we get
\[
J_2 \leq C \sum_{i=0}^{\infty} 2^{i(r-1)} l(2^i) \max_{2^i \leq n < 2^{i+1}} [n^{-q/t} E \| S'_n \| - E \| S'_n \| | \| n \| ] < \infty.
\]
The proof of (b) and (c) is similar to that (c) and (d) in Theorem 1, respectively, so is omitted here.

Acknowledgement I would like to express my thanks to Prof. Hu Dihe for his encouragement and valuable help.

References
1 Bai Z D, Su C. The complete convergence for partial sums of i.i.d.random variables. Sci Sinica (Ser.A), 1985,28: 1261-1277